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New similarity reductions and Painlevé analysis for the symmetric regularised long wave and modified Benjamin–Bona–Mahoney equations

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Abstract. In this paper we discuss similarity reductions and Painlevé analysis for the symmetric regularised long wave and modified Benjamin–Bona–Mahoney equations, both of which arise in several physical applications including shallow water waves. Both equations are thought to non-integrable (i.e. not solvable by inverse scattering) since numerical studies show that the interaction of solitary waves is inelastic. In particular, we determine some new similarity reductions of the symmetric regularised long wave equation. These new similarity reductions are *not* obtainable using the classical Lie group method for finding group-invariant solutions of partial differential equations; they are determined using a new and direct method which involves no group theoretical techniques. It is shown that every similarity reduction of both the symmetric regularised long wave and modified Benjamin–Bona–Mahoney equations obtained using the classical Lie group method reduces the associated partial differential equation to an ordinary differential equation of Painlevé type; whereas the new similarity solution of the symmetric regularised long wave equation reduces it to an ordinary differential equation which is not of Painlevé type. It is also shown that neither the symmetric regularised long wave equation nor the modified Benjamin–Bona–Mahoney equation possesses the Painlevé property for partial differential equations as defined by Weiss *et al.*

1. Introduction

In this paper we discuss similarity reductions and Painlevé analysis for the symmetric regularised long wave (SRLW) equation

$$u_{tt} + au_{xx} + \tfrac{1}{2}b(u^2)_{xt} + cu_{xxtt} = 0 \tag{1.1}$$

and the modified Benjamin–Bona–Mahoney (MBBM) equation

$$u_t + au_x + bu^2u_x + cu_{xxt} = 0 \tag{1.2}$$

where subscripts denote differentiation and a, b, c are constants.

The Benjamin–Bona–Mahoney (BBM) equation (which is sometimes called the regularised long wave (RLW) equation)

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.3}$$

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was proposed by Benjamin *et al* (1972) as an alternative to the celebrated Korteweg–de Vries (KdV) equation (Korteweg and de Vries 1895)

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.4}$$

which is a soliton equation solvable by inverse scattering (Gardner *et al* 1967), for the description of long waves in shallow water. Similarly, the MBBM equation (1.3) may be viewed as an alternative to the modified KdV (MKdV) equation

$$u_t + u_x + u^2u_x + u_{xxx} = 0 \tag{1.5}$$

which also is a soliton equation solvable by inverse scattering (Wadati 1972, Ablowitz *et al* 1974).

The SRLW equation (1.1) was named by Seyler and Fenstermacher (1984) due to its similarity to the BBM equation (1.3) together with its explicit symmetry with respect to the x and t derivatives and arises in several physical applications including ion sound waves in a plasma (Bogolubsky 1977, Makhankov 1978, Ogino and Takenda 1976, Seyler and Fenstermacher 1984). It may be viewed as alternative to the Boussinesq equation (Boussinesq 1871, 1872)

$$u_{tt} + au_{xx} + b(u^2)_{xx} + cu_{xxxx} = 0 \tag{1.6}$$

where a, b, c are constants, which also arises in several physical applications, and like the KdV equation (1.4) and the MKdV equation (1.5) is a soliton equation solvable by inverse scattering (Zakharov 1974, Ablowitz and Haberman 1975, Caudrey 1980, 1982, Deift *et al* 1982).

The inverse scattering method was originally developed by Gardner *et al* (1967) to solve the Cauchy problem for the KdV equation (1.4) (for initial data on the infinite line which decays sufficiently rapidly). In effect, this method reduces the solution of the non-linear partial differential equation to that of a linear integral equation, and the partial differential equation is then said to be *completely integrable*. Such equations all seem to possess several remarkable properties including elastically interacting soliton solutions, the existence of infinitely many independent conservation laws and symmetries, Bäcklund transformations, Lax representation, the Painlevé property, etc (cf Ablowitz and Segur 1981). However, the precise relationship between all these properties has yet to be fully established.

It is known that the SRLW equation (1.1) and the MBBM equation (1.2) possess solitary wave solutions

$$u(x,t) = -\frac{3(\gamma^2+a)}{2\gamma b}\,\mathrm{sech}^2\left[\frac{1}{2}\left(\frac{\gamma^2+a}{-\gamma^2 c}\right)^{1/2}(x \pm \gamma t) + x_0\right]$$

$$u(x,t) = -\left(\frac{6(\gamma+a)}{-b}\right)^{1/2}\mathrm{sech}\left[\left(\frac{\gamma+a}{-\gamma c}\right)^{1/2}(x + \gamma t) + x_0\right]$$

where γ, x_0 are constants, respectively. However, both equations are thought *not* to be completely integrable since numerical evidence suggests that the interaction of two solitary waves is inelastic and so they are not solitons (cf Bogolubsky 1977, Makhankov 1978, Seyler and Fenstermacher 1984). Additionally the SRLW equation possesses only three independent polynomial conservation laws (Seyler and Fenstermacher 1984).

To study similarity reductions of the SRLW and MBBM equations, we assume that $a = 1$, $b = 1$, $c = \pm 1$, since the equations

$$u_{tt} + u_{xx} + \tfrac{1}{2}(u^2)_{xt} \pm u_{xxtt} = 0 \tag{1.7}$$

$$u_t + u_x + u^2 u_x \pm u_{xxt} = 0 \tag{1.8}$$

are equivalent to equations (1.1) and (1.2), after a suitable rescaling and translation of the variables. If the quantities in the equations are to be interpreted as real, then the sign matters and we choose the plus signs from here on for convenience (and leave to the reader the trivial modifications required for the other sign). However, if the quantities are interpreted as complex, then the sign does not matter and our analysis is complete.

The classical method of finding similarity reductions of a given partial differential equation is the Lie group method of infinitesimal transformations (sometimes called the method of group-invariant solutions), originally due to Lie (1891)—for recent descriptions of this method see Bluman and Cole (1974), Olver (1986), Ovsiannikov (1982) and Winternitz (1983). Although the method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually. Symbolic manipulation programs have been developed, both in MACSYMA (Rosenau and Schwarzmeier 1979, Champagne and Winternitz 1985) and REDUCE (Schwarz 1985), to facilitate the determination of similarity reductions using the Lie group method. (See Schwarz (1988) for a recent review on the use of computer algebra to find symmetries of differential equations.) Bluman and Cole (1969) proposed a generalisation of Lie's method called the *non-classical method of group-invariant solutions*, which itself has recently been generalised by Olver and Rosenau (1986, 1987).

All these three methods determine Lie point symmetries of a given partial differential equation since the transformations depend only on the independent and dependent variables. Another common characteristic of these methods for finding similarity reductions of a given partial differential equation is the use of group theory.

In this paper we use the direct method of deriving similarity reductions of partial differential equations recently developed by Clarkson and Kruskal (1989). The unusual characteristic about this method in comparison to the others mentioned above is that it involves *no* use of group theory. It has been used to obtain new similarity reductions of the Boussinesq equation (1.6) (Clarkson and Kruskal 1989) and the modified Boussinesq equation

$$q_{tt} - q_t q_{xx} - \tfrac{1}{2} q_x^2 q_{xx} + q_{xxxx} = 0$$

(Clarkson 1989). In this paper we use the method to obtain similarity reductions of the SRLW equation (1.7) and the MBBM equation (1.8). The basic idea is to seek a solution of a given partial differential equation in the form

$$u(x, t) = U\,(x, t, w\,(z(x, t))) \tag{1.9}$$

which is the most general form for a similarity reduction (cf Bluman and Cole 1974). Then we require that substitution of (1.9) into the partial differential equation yields an ordinary differential equation for $w(z)$. This imposes conditions upon U, z and their derivatives which enable one to solve for U and z.

The Painlevé conjecture (or Painlevé ODE test), as formulated by Ablowitz *et al* (1978, 1980) and Hastings and McLeod (1980), asserts that every ordinary differential equation which arises as a similarity reduction of a completely integrable partial differential equation is of Painlevé type (i.e. its solutions have no movable singularities other than poles), though perhaps only after a transformation of variables. Subsequently, Weiss *et al* (1983) developed the Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given partial differential equation, without having to study any similarity reductions (which might not exist anyway). A partial differential equation is said to pass the Painlevé PDE test if all its solutions (both general and singular) are 'single-valued' in the neighbourhood of arbitrary non-characteristic movable singularity manifolds (this is explained further in §5 below).

The outline of this paper is as follows: in §2 we describe the previously known similarity reductions of the SRLW and MBBM equations; in §§3 and 4 we use this direct method to obtain similarity reductions of the SRLW and MBBM equations, respectively; in §5 we discuss the application of the Painlevé tests to the SRLW and MBBM equations; and in §6 we discuss our results.

2. Classical similarity reductions

Here we derive the classical similarity reductions of the SRLW and MBBM equations (1.7), (1.8) using the Lie group method as given by Bluman and Cole (1974). Consider the one-parameter (ε) Lie group of infinitesimal transformations in (x, t, u) given by

$$\xi = x + \varepsilon X(x,t,u) + \mathrm{O}(\varepsilon^2) \tag{2.1a}$$

$$\tau = t + \varepsilon T(x,t,u) + \mathrm{O}(\varepsilon^2) \tag{2.1b}$$

$$\eta = u + \varepsilon U(x,t,u) + \mathrm{O}(\varepsilon^2) \tag{2.1c}$$

$$\eta_\xi = u_x + \varepsilon U^x + \mathrm{O}(\varepsilon^2) \tag{2.2a}$$

$$\eta_\tau = u_t + \varepsilon U^t + \mathrm{O}(\varepsilon^2) \tag{2.2b}$$

$$\eta_{\xi\xi} = u_{xx} + \varepsilon U^{xx} + \mathrm{O}(\varepsilon^2) \tag{2.2c}$$

$$\eta_{\xi\tau} = u_{xt} + \varepsilon U^{xt} + \mathrm{O}(\varepsilon^2) \tag{2.2d}$$

$$\eta_{\tau\tau} = u_{tt} + \varepsilon U^{tt} + \mathrm{O}(\varepsilon^2) \tag{2.2e}$$

$$\eta_{\xi\xi\tau} = u_{xxt} + \varepsilon U^{xxt} + \mathrm{O}(\varepsilon^2) \tag{2.2f}$$

$$\eta_{\xi\xi\tau\tau} = u_{xxtt} + \varepsilon U^{xxtt} + \mathrm{O}(\varepsilon^2) \tag{2.2g}$$

where the infinitesimals U^x, U^t, U^{xx}, U^{xt}, U^{tt}, U^{xxt}, U^{xxtt} are determined from equations (2.1) (cf Bluman and Cole 1974). The equations (1.7), (1.8) are invariant under this transformation if $\eta(\xi, \tau)$ satisfies the same equation as $u(x, t)$. Substituting (2.1), (2.2) into the SRLW and MBBM equations for $\eta(\xi, \tau)$, then to first order in ε we have

$$U^{tt} + U^{xx} + uU^{xt} + u_{xt}U + u_xU^t + u_tU^x + U^{xxtt} = 0. \tag{2.3a}$$

$$U^t + U^x + u^2U^x + 2uu_xU + U^{xxt} = 0. \tag{2.3b}$$

respectively. Conditions on the infinitesimals $X(x,t,u)$, $T(x,t,u)$, $U(x,t,u)$ are determined by collecting coefficients of like derivative terms in u and equating them to zero in equations (2.3). Solving these determining equations for both the SRLW equation (1.7) and the MBBM equation (1.8) yields

$$X=\alpha \qquad T=\beta \qquad U=0 \tag{2.4}$$

where α, β, γ are arbitrary constants (cf Clarkson 1983, Rosenau and Schwarzmeier 1986).

Similarity reductions are obtained by solving the characteristic equations

$$\frac{\mathrm{d}x}{X(x,t,u)}=\frac{\mathrm{d}t}{T(x,t,u)}=\frac{\mathrm{d}u}{U(x,t,u)}. \tag{2.5}$$

Hence for both the the SRLW equation (1.7) and MBBM equation (1.8), the only similarity reduction obtained by the classical Lie method is the travelling wave solution

$$u(x,t)=f(z) \qquad z=\beta x-\alpha t \tag{2.6}$$

where for the SRLW equation, $f(z)$ satisfies

$$(\beta^2+\alpha^2)f-\tfrac{1}{2}\alpha\beta f^2+\alpha^2\beta^2 f''=Az+B \tag{2.7}$$

and for the MBBM equation, $f(z)$ satisfies

$$\beta^2\alpha(f')^2-\tfrac{1}{6}\alpha f^4+(\beta-\alpha)f^2=Af+B \tag{2.8}$$

with $' := \mathrm{d}/\mathrm{d}z$ and A, B arbitrary constants of integration. If $\alpha\beta\neq 0$, then the solution $f(z)$ of equation (2.7) is expressible in terms of either Weierstrass elliptic functions (cf Whittaker and Watson 1927) or the first Painlevé equation (cf Ince 1956)

$$w''=6w^2+z \tag{2.9}$$

depending upon the choice of constants; whilst equation (2.8) is solvable in terms of Jacobian elliptic functions (cf Whittaker and Watson 1927). Therefore every similarity reduction of either the SRLW equation or the MBBM equation obtained using the classical Lie method of infinitesimal transformations, reduces the partial differential equation to an ordinary differential equation of Painlevé type.

3. New similarity reductions of SRLW equation

In this section we seek solutions of the SRLW equation (1.7) in the form

$$u(x,t)=U\,(x,t,w\,(z(x,t))). \tag{3.1}$$

Actually, as shown below, it suffices to seek solutions of the SRLW equation in the form

$$u(x,t)=\alpha(x,t)+\beta(x,t)w\,(z(x,t)) \tag{3.2}$$

where $\alpha(x,t)$, $\beta(x,t)$, $z(x,t)$ are to be determined. However, before doing this we make some remarks about this direct method of seeking similarity reductions (using the simplified ansatz (3.2)).

Remark 3.1. We substitute (3.2) into the partial differential equation and then require that the resulting equation is an ordinary differential equation for $w(z)$, so it is necessary that the ratios of different derivatives and powers of $w(z)$ be functions of z only. This gives a set of conditions for $\alpha(x,t)$, $\beta(x,t)$, $z(x,t)$ in the form of an overdetermined system of equations, any solution of which yields a similarity reduction. (These conditions are both *necessary and sufficient* for (3.2) to reduce the partial differential equation for $u(x,t)$ to an ordinary differential equation for $w(z)$.)

Remark 3.2. We use the coefficient of w'''' (i.e. $\beta z_x^2 z_t^2$), provided that $z_x z_t \not\equiv 0$, as the normalising coefficient and therefore require that the other coefficients are necessarily of the form $\beta z_x^2 z_t^2 \Gamma(z)$, where Γ is a function of z to be determined.

Remark 3.3. Whenever we use an upper case Greek letter to denote a function (e.g. $\Gamma(z)$), then this is a function, to be determined, upon which we can perform any mathematical operation (e.g. differentiation, integration, logarithm, exponentiation, taking powers, rescaling, etc) and then also call the resulting function $\Gamma(z)$, without loss of generality (e.g. the differential of $\Gamma(z)$ will be called $\Gamma(z)$).

Remark 3.4. There are three freedoms in the determination of α, β, z which we can exploit, without loss of generality (these are valuable in keeping the method manageable):

(*a*) if $\alpha(x,t)$ is of the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Gamma(z)$ then we can assume that $\Gamma \equiv 0$ (make the transformation $w(z) \to w(z) - \Gamma(z)$);

(*b*) if $\beta(x,t)$ is of the form $\beta = \beta_0(x,t)\Gamma(z)$, then we can assume that $\Gamma \equiv 1$ (make the transformation $w(z) \to w(z)/\Gamma(z)$);

(*c*) if $z(x,t)$ is defined by an equation of the form $\Gamma(z) = z_0(x,t)$, where $\Gamma(z)$ is any invertible function, then we can assume that $\Gamma \equiv z$ (make the transformation $z \to \Gamma^{-1}(z)$, where Γ^{-1} is the inverse of Γ).

Substituting (3.1) into (1.7) yields

$$\begin{aligned}
U_{tt} &+ 2U_{tw}w'z_t + U_{ww}(w')^2 z_t^2 + U_w(w'' z_t^2 + w' z_{tt}) \\
&+ U_{xx} + 2U_{xw}w'z_x + U_{ww}(w')^2 z_x^2 + U_w(w'' z_x^2 + w' z_{xx}) \\
&+ U\left[U_{xt} + (U_{tw}z_x + U_{xw}z_t)w' + U_{ww}(w')^2 z_x z_t + U_w(w'' z_x z_t + w' z_{xt})\right] \\
&+ \left[U_x + U_w w' z_x\right]\left[U_t + U_w w' z_t\right] + U_{xxtt} + 2\left[U_{xxtw}z_t + U_{xttw}z_x\right] w' \\
&+ \left[U_{xxww}z_t^2 + 4U_{xtww}z_x z_t + U_{ttww}z_x^2\right](w')^2 \\
&+ 2\left[U_{xwww}z_x z_t^2 + U_{twww}z_x^2 z_t\right](w')^3 \\
&+ U_{wwww}(w')^4 z_x^2 z_t^2 + U_{xxw}(w'' z_t^2 + w' z_{tt}) + 4U_{xtw}(w'' z_x z_t + w' z_{xt}) \\
&+ U_{ttw}(w'' z_x^2 + w' z_{xx}) + U_{xww}\left[6w'w'' z_x^2 z_t + (w')^2(4z_t z_{xt} + 2z_x z_{tt})\right] \\
&+ U_{tww}\left[6w'w'' z_x z_t^2 + (w')^2(4z_x z_{xt} + 2z_t z_{xx})\right] \\
&+ U_{www}\left[6(w')^2 w'' z_x^2 z_t^2 + (w')^3(4z_x z_t z_{xt} + z_x^2 z_{tt} + z_t^2 z_{xx})\right]
\end{aligned}$$

$$
\begin{aligned}
&+ U_{xw}\left[2w'''z_xz_t^2 + w''(4z_tz_{xt} + 2z_xz_{tt}) + w'z_{xtt}\right]\\
&+ U_{tw}\left[2w'''z_x^2z_t + w''(4z_xz_{xt} + 2z_tz_{xx}) + w'z_{xxt}\right]\\
&+ U_{ww}\left[(4w'w''' + 3(w'')^2)z_x^2z_t^2 + w'w''(12z_xz_tz_{xt} + 3z_x^2z_{tt} + 3z_t^2z_{xx})\right.\\
&\left.+ (w')^2(z_{xx}z_{tt} + 2z_{xt}^2 + 2z_xz_{xtt} + 2z_tz_{xxt})\right]\\
&+ U_w\left[w''''z_x^2z_t^2 + w'''(4z_xz_tz_{xt} + z_x^2z_{tt} + z_t^2z_{xx})\right.\\
&\left.+ w''(z_{xx}z_{tt} + 2z_{xt}^2 + 2z_xz_{xtt} + 2z_tz_{xxt}) + w'z_{xxtt}\right] = 0
\end{aligned}
\tag{3.3}
$$

where $' := \mathrm{d}/\mathrm{d}z$. In order that this is an ordinary differential equation for $w(z)$, then the ratios of different derivatives of $w(z)$ have to be functions of w and z. If we use the coefficient of w'''' (i.e. $U_wz_x^2z_t^2$), as the normalising coefficient (assuming that $z_xz_t \not\equiv 0$—we discuss the case when $z_xz_t \equiv 0$ below), then the coefficients of $w'w'''$ and $(w'')^2$ require that

$$U_wz_x^2z_t^2\Gamma(w,z) = U_{ww}z_x^2z_t^2$$

where $\Gamma(w,z)$ is a function to be determined. Hence

$$\Gamma(w,z) = U_{ww}/U_w$$

which upon integration yields

$$U_w = \Theta(x,t)\Gamma(w,z) \tag{3.4}$$

with $\Theta(x,t)$ a function of integration (recall remark 3.3 above). Integrating again yields

$$U(x,t,w) = \Theta(x,t)\Gamma(w,z) + \Phi(x,t)$$

with $\Phi(x,t)$ another function of integration. Therefore it is sufficient to seek similarity reductions of the SRLW equation (1.7) in the form (3.2) (in the case when $z_xz_t \not\equiv 0$).

Substituting (3.2) into (1.7) and collecting coefficients of like derivatives and powers of $w(z)$ yields

$$
\begin{aligned}
&\beta z_x^2z_t^2w'''' + \left[(2\beta_xz_x + \beta z_{xx})z_t^2 + (2\beta_tz_t + \beta z_{tt})z_x^2 + 4\beta z_xz_tz_{xt}\right]w'''\\
&\quad + \left[\beta_{xx}z_t^2 + 4\beta_{xt}z_xz_t + \beta_{tt}z_x^2 + \beta_x(2z_xz_{tt} + 4z_xz_{xt}) + \beta_t(2z_tz_{xx} + 4z_tz_{xt})\right.\\
&\quad \left.+ \beta(z_{xx}z_{tt} + 2z_{xt}^2 + 2z_xz_{xtt} + 2z_xz_{xxt}) + \beta(z_t^2 + z_x^2) + \alpha\beta z_xz_t\right]w''\\
&\quad + \left[2\beta_{xxt}z_t + 2\beta_{xtt}z_x + \beta_{xx}z_{tt} + 4\beta_{xt}z_{xt} + \beta_{tt}z_{xx}\right.\\
&\quad + 2\beta_xz_{xtt} + 2\beta_tz_{xxt} + \beta z_{xxtt} + (2\beta_tz_t + \beta z_{tt}) + (2\beta_xz_x + \beta z_{xx})\\
&\quad \left.+ \alpha(\beta_xz_t + \beta_tz_x + \beta z_{xt}) + \beta(\alpha_xz_t + \alpha_tz_x)\right]w'\\
&\quad + \left[\beta_{xxtt} + \beta_{tt} + \beta_{xx} + \alpha\beta_{xt} + \alpha_{xt}\beta + \alpha_x\beta_t + \alpha_t\beta_x\right]w\\
&\quad + (\beta\beta_{xt} + \beta_x\beta_t)w^2 + \left[\beta^2z_{xt} + 2\beta(\beta_xz_t + \beta_tz_x)\right]ww' + \beta^2z_xz_t\left[ww'' + (w')^2\right]\\
&\quad + \alpha_{xxtt} + \alpha_{tt} + \alpha_{xx} + \alpha\alpha_{xt} + \alpha_x\alpha_t = 0
\end{aligned}
\tag{3.5}
$$

with $' := \mathrm{d}/\mathrm{d}z$.

The coefficients of ww'' and $(w')^2$ yield the common constraint

$$\beta z_x^2 z_t^2 \Gamma(z) = \beta^2 z_x z_t$$

where $\Gamma(z)$ is a function to be determined. Using the freedom mentioned in remark 3.4(b),

$$\beta = z_x z_t. \tag{3.6}$$

The coefficients of ww' and w''', after using (3.6), yield the constraints

$$z_x^2 z_t^2 \Gamma_1(z) = 2z_{xx}z_t^2 + 2z_{tt}z_x^2 + 5z_x z_t z_{xt} \tag{3.7}$$

$$z_x^2 z_t^2 \Gamma_2(z) = 3z_{xx}z_t^2 + 3z_{tt}z_x^2 + 8z_x z_t z_{xt} \tag{3.8}$$

respectively, where $\Gamma_1(z)$, $\Gamma_2(z)$ are to be determined. Hence from these equations we have

$$z_x \Gamma(z) + z_{xt}/z_t = 0$$

where $\Gamma(z) = 2\Gamma_2(z) - 3\Gamma_1(z)$. Integrating with respect to x yields

$$z_t \Gamma(z) = \Sigma(t)$$

with $\Sigma(t)$ a function of integration (recall remark 3.3). Integrating with respect to t gives

$$\Gamma(z) = \Theta(x) + \Sigma(t)$$

with $\Theta(x)$ another function of integration. Using the freedom mentioned in remark 3.4(c), we have

$$z(x,t) = \theta(x) + \sigma(t) \tag{3.9}$$

where $\theta(x)$ and $\sigma(t)$ are to be determined. Hence from equation (3.6) we also have

$$\beta(x,t) = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}. \tag{3.10}$$

The coefficients of w^2 and ww' (or w'''), after using (3.9)–(3.10), yield the constraints

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \Gamma_3(z) = \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \Gamma_4(z) = \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2$$

respectively, where $\Gamma_3(z)$ and $\Gamma_4(z)$ are to be determined. Therefore

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = c_1\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \qquad \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} = c_2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \tag{3.11}$$

with c_1, c_2 arbitrary constants.

The coefficient of w'', after using (3.9)–(3.11), yields the constraint to

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 [\Gamma(z) - 2c_1^2 - 2c_2^2 - 5c_1c_2] = \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2\right] + \alpha\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

where $\Gamma_5(z)$ is to be determined. Without loss of generality, we choose

$$\Gamma_5(z) = 2c_1^2 + 2c_2^2 + 5c_1c_2$$

(recall remark 3.4(*a*)), and therefore $\alpha(x,t)$ is defined by

$$\alpha(x,t)\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2\right]. \tag{3.12}$$

Solving (3.11) gives

$$\theta(x) = \begin{cases} -c_1^{-1}\ln(x+x_0) + x_1 & \text{if } c_1 \neq 0 \\ c_3(x+x_0) & \text{if } c_1 = 0 \end{cases} \tag{3.13a}$$

$$\sigma(t) = \begin{cases} -c_2^{-1}\ln(t+t_0) + t_1 & \text{if } c_2 \neq 0 \\ c_4(t+t_0) & \text{if } c_2 = 0 \end{cases} \tag{3.13b}$$

with c_3, c_4, x_0, x_1, t_0, t_1 arbitrary constants (we set $x_0 = 0$, $x_1 = 0$, $t_0 = 0$, $t_1 = 0$ without loss of generality).

The coefficients of w', w and 1 (i.e. the term not involving either w or its derivatives), in equation (3.5), after using (3.9)–(3.12), yield the constraints

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \Gamma_6(z) = c_1\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + c_2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + 6c_1c_2(c_1+c_2)\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \tag{3.14a}$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \Gamma_7(z) = c_1^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + c_2^2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + 2c_1^2c_2^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \tag{3.14b}$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \Gamma_8(z) = c_1\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + c_2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 \tag{3.14c}$$

respectively, where $\Gamma_6(z)$, $\Gamma_7(z)$ and $\Gamma_8(z)$ are to be determined. Suppose that $c_1c_2 \neq 0$, then from (3.13), (3.14*a*) simplifies to

$$\Gamma_6\left(\frac{\ln(xt)}{c_1c_2}\right) = c_1c_2(c_1x^2 + c_2t^2) + 6c_1c_2(c_1+c_2) \tag{3.15}$$

which is impossible (i.e. there exists no function $\Gamma_6(z)$ such that this holds). Therefore $c_1c_2 = 0$ is a necessary (and sufficient) condition for equations (3.13) and (3.14) to be compatible, and from equations (3.14) we see that

$$\Gamma_6 = \frac{c_1}{c_4^2} + \frac{c_2}{c_3^2} \qquad \Gamma_7 = \frac{c_1^2}{c_4^2} + \frac{c_2^2}{c_3^2} \qquad \Gamma_8 = \frac{c_1}{c_4^2} + \frac{c_2}{c_3^2}.$$

All the (necessary and sufficient) conditions obtained by the requirement that the ratios of coefficients of powers of w and its derivatives in equation (3.5) are functions of z only, have now been satisfied. Therefore the SRLW equation (1.7) possesses a similarity reduction of the form $u(x,t) = U(x,t,w(z))$, for some U and z, if and only if it has the form

$$u(x,t) = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\,w(z) - \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2\right]\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^{-1} \tag{3.16a}$$

with

$$z(x,t) = \theta(x) + \sigma(t) \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \not\equiv 0 \tag{3.16b}$$

where $\theta(x)$ and $\sigma(t)$ are given by equations (3.13), provided that $c_1c_2 = 0$ and $c_3c_4 \neq 0$, $w(z)$ satisfies the ordinary differential equation

$$\begin{aligned} w'''' + ww'' + (w')^2 + 3(c_1+c_2)w''' + 2(c_1^2+c_2^2)w'' + \left(\frac{c_1}{c_4^2} + \frac{c_2}{c_3^2}\right)w' \\ + 2\left(\frac{c_1^2}{c_4^2} + \frac{c_2^2}{c_3^2}\right)w + 2(c_1+c_2)ww' - 2\left(\frac{c_1}{c_4^2} + \frac{c_2}{c_3^2}\right) = 0 \end{aligned} \tag{3.17}$$

with $' = \mathrm{d}/\mathrm{d}z$.

There are three cases to consider.

Case 1. $c_1 = 0$, $c_2 = 0$. We obtain the similarity reduction

$$u(x,t) = c_3c_4w_1(z) - \left(\frac{c_3}{c_4} + \frac{c_4}{c_3}\right) \qquad z = c_3x + c_4t. \tag{3.18}$$

(Since we have assumed that $z_xz_t \not\equiv 0$, then necessarily $c_3c_4 \neq 0$.) Making the transformation

$$v_1(z) = c_3c_4w_1(z) - \left(\frac{c_3}{c_4} + \frac{c_4}{c_3}\right) \tag{3.19}$$

$v_1(z)$ satisfies

$$(c_3^2 + c_4^2)v_1 - \tfrac{1}{2}c_3c_4v_1^2 + c_3^2c_4^2\frac{\mathrm{d}^2v_1}{\mathrm{d}z^2} = Az + B \tag{3.20}$$

with A, B constants, which is the same equation as (2.7). This is the travelling wave solution which we also obtained in §2 using the classical Lie group method. Since $c_3c_4 \neq 0$, the solution of equation (3.20) is expressible either in terms of either Weierstrass elliptic functions or the first Painlevé equation (2.9) (depending upon the choice of constants), and so is meromorphic and thus equation (3.20) is of Painlevé type.

Case 2. $c_1 = 0$, $c_2 \neq 0$. We obtain the similarity reduction

$$u(x,t) = -\frac{c_3}{c_2t}\,w_2(z) + c_2c_3t + \frac{1}{c_2c_3t} \qquad z = c_3x - \frac{\ln t}{c_2}. \tag{3.21}$$

Since necessarily $c_3 \neq 0$, then we set $c_3 = 1$, without loss of generality. After making the transformation $w_2(z) = 1 - c_2 v_2(z)$, $v_2(z)$ satisfies

$$\frac{\mathrm{d}^4 v_2}{\mathrm{d}z^4} + 3c_2 \frac{\mathrm{d}^3 v_2}{\mathrm{d}z^3} + (2c_2^2 + 1)\frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2} + 3c_2 \frac{\mathrm{d}v_2}{\mathrm{d}z} + 2c_2^2 v_2 - 2c_2^2 v_2 \frac{\mathrm{d}v_2}{\mathrm{d}z} - c_2\left[v_2 \frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2} + \left(\frac{\mathrm{d}v_2}{\mathrm{d}z}\right)^2\right] = 0. \tag{3.22}$$

In §5 below, we show that there exist solutions of this equation which have movable logarithmic branch points and so it is *not* of Painlevé type.

The similarity reduction

$$u(x,t) = t^{-1} w(z) + \lambda t \qquad z = x - \frac{\ln t}{\lambda} \tag{3.23}$$

with λ ($\not\equiv 0$) a constant, is a new similarity reduction of the SRLW equation (1.7). A one-parameter group associated with this similarity reduction is given by

$$(x,t,u) \to \left(x + \gamma/\lambda, t\,\mathrm{e}^{\gamma}, u\,\mathrm{e}^{-\gamma} + \lambda t(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma})\right) \tag{3.24a}$$

for which the associated infinitesimals are

$$X(x,t,u) = \lambda^{-1} \qquad T(x,t,u) = t \qquad U(x,t,u) = 2\lambda t - u \tag{3.24b}$$

which are clearly *not* a special case of the infinitesimals obtained using the classical Lie group method for the SRLW equation (i.e. equation (2.4)).

Case 3. $c_1 \neq 0$, $c_2 = 0$. This case is the same as case 2 above with x and t interchanged. Setting $c_4 = 1$, we obtain the similarity reduction

$$u(x,t) = x^{-1} v_3(z) + c_1 x \qquad z = t - \frac{\ln x}{c_1}$$

where $v_3(z)$ satisfies the same equation as $v_2(z)$ (i.e. equation (3.22)), with c_2 replaced by c_1.

Now we discuss the case when $z_x z_t \equiv 0$. Since the SRLW equation is symmetric in x and t, it is suffices to seek solutions in the form

$$u(x,t) = U(x,t,y(x)). \tag{3.25}$$

Substituting this into the SRLW equation (1.7) yields

$$U_{tt} + U_{xx} + UU_{xt} + U_x U_t + U_{xxtt} + (U_y + U_{tty})y'' + 2(U_{xy} + U_{xtty} + UU_{ty} + U_t U_y)y' + (U_y + U_{tty})(y')^2 = 0 \tag{3.26}$$

where $'$:= $\mathrm{d}/\mathrm{d}x$. By considering the coefficients of the y'' and $(y')^2$ terms, it is easily shown that it is sufficient to seek solutions in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)\,y(x). \tag{3.27}$$

Substituting this into (1.7) yields

$$\begin{aligned}\alpha_{tt} + \alpha_{xx} + \alpha\alpha_{xt} + \alpha_x\alpha_t + \alpha_{xxtt} + (\beta\beta_{xt} + \beta_x\beta_t)y^2 + 2\beta\beta_t yy' \\ + (\beta_{tt} + \beta_{xx} + \alpha_{xt}\beta + \alpha\beta_{xt} + \alpha_x\beta_t + \alpha_t\beta_x + \beta_{xxtt})y \\ + (2\beta_x + \alpha\beta_t + \alpha_t\beta + 2\beta_{xtt})y' + (\beta + \beta_{tt})y'' = 0.\end{aligned} \tag{3.28}$$

This is an ordinary differential equation for $y(x)$ if the ratios of coefficients of different powers and derivatives of y are functions of x only. There are three cases to consider (since the calculations are similar to those done in the more general case above, details are omitted).

Case 1. $\beta_t \equiv 0$. Without loss of generality we assume that $\beta \equiv 1$ (recall remark 3.4(*b*)). Then it is easily shown that there are solutions of the SRLW equation of the form

$$u(x, t) = y(x) + t\phi(x)$$

where $y(x)$ and $\phi(x)$ satisfy the equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \phi y = A \tag{3.29a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} + \phi^2 = B \tag{3.29b}$$

with A, B arbitrary constants. Equation (3.29*b*) has the following solutions:

$$\begin{aligned} &\phi_1(x) = k\tanh[k(x + x_0)] && \text{if } B = k^2, k > 0 \\ &\phi_2(x) = k\coth[k(x + x_0)] && \text{if } B = k^2, k > 0 \\ &\phi_3(x) = \pm k && \text{if } B = k^2, k > 0 \\ &\phi_4(x) = (x + x_0)^{-1} && \text{if } B = 0 \\ &\phi_5(x) = -k\tan[k(x + x_0)] && \text{if } B = -k^2, k > 0 \end{aligned}$$

where x_0 are constants. (We remark that ϕ_1, ϕ_2 and ϕ_5 are related by complex transformations, ϕ_3 can be obtained by letting $x_0 \to \pm\infty$ in either ϕ_1 or ϕ_2, and ϕ_4 can be obtained by letting $k \to 0$ in either ϕ_1, ϕ_2 or ϕ_5.) Consequently we obtain the following special solutions of the SRLW equation:

$$u_1(x, t) = C\,\mathrm{sech}[k(x + x_0)] + k(t + t_0)\tanh[k(x + x_0)] \tag{3.30a}$$

$$u_2(x, t) = C\,\mathrm{cosech}[k(x + x_0)] + k(t + t_0)\coth[k(x + x_0)] \tag{3.30b}$$

$$u_3(x, t) = A\exp(\mp kx) \pm k(t + t_0) \tag{3.30c}$$

$$u_4(x, t) = \tfrac{1}{2}A(x + x_0) + (t + C)/(x + x_0) \tag{3.30d}$$

$$u_5(x, t) = C\sec[k(x + x_0)] - k(t - t_0)\tan[k(x + x_0)] \tag{3.30e}$$

with x_0, C arbitrary constants and $t_0 := A/k^2$.

Case 2. $\beta_t \not\equiv 0$, $\beta_x \equiv 0$. In this case we obtain the special solution

$$u_5(x,t) = \frac{x+x_1}{t+t_1} + \frac{x+x_2}{t+t_2}$$

with x_1, x_2, t_1, t_2 arbitrary constants.

Case 3. $\beta_t \not\equiv 0$, $\beta_x \not\equiv 0$. After some manipulation it can be shown that $\beta(x,t)$ has the form

$$\beta(x,t) = t\theta(x) + \sigma(x)$$

where $\theta(x)$, $\sigma(x)$ are to be determined. Furthermore, we can assume, without loss of generality that $\theta(x) \equiv 1$ (recall remark 3.4(*b*)). Then necessarily $\mathrm{d}\sigma/\mathrm{d}x = 0$, and so $\beta = t + t_0$, with t_0 a constant, which contradicts the initial assumption that $\beta_x \not\equiv 0$. Therefore there are no special solutions of the SRLW equation in this case.

Since the SRLW equation is symmetric in x and t, then we obtain further special solutions by interchanging x and t in equations (3.30). We note that all the special solutions of the SRLW equation are meromorphic with respect to x and t.

4. Similarity reductions of the MBBM equation

In this section we determine similarity reductions of the MBBM equation. As for the SRLW equation in §3, it suffices to seek similarity reductions in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w\,(z(x,t)) \tag{4.1}$$

provided $z_x z_t \not\equiv 0$. Substituting this into (1.8) and collecting coefficients yields

$$\begin{aligned}\beta z_x^2 z_t w''' &+ [2\beta_x z_x z_t + \beta_t z_x^2 + \beta(z_t z_{xx} + 2z_x z_{xt})]w'' \\ &+ [\beta_{xx} z_t + 2\beta_{xt} z_x + 2\beta_x z_{xt} + \beta_t z_{xx} + \beta z_{xxt} + \beta(z_t + z_x) + \alpha^2 \beta z_x]w' \\ &+ [\beta_{xxt} + \beta_t + \beta_x + 2\alpha\alpha_x\beta + \alpha^2\beta_x]w + [\alpha_x\beta^2 + 2\alpha\beta\beta_x]w^2 + 2\alpha\beta^2 z_x ww' \\ &+ \beta^2\beta_x w^3 + \beta^3 z_x w^2 w' + \alpha_t + \alpha_x + \alpha^2\alpha_x + \alpha_{xxt} = 0\end{aligned} \tag{4.2}$$

with $' := \mathrm{d}/\mathrm{d}z$.

We use the coefficient of w''' as the normalising coefficient, then for (4.2) to be an ordinary differential equation for $w(z)$, the coefficients of w^2w' and w^3 yield the constraints

$$\beta z_x^2 z_t \Gamma_1(z) = \beta^3 z_x \tag{4.3a}$$

$$\beta z_x^2 z_t \Gamma_2(z) = \beta^2 \beta_x \tag{4.3b}$$

where $\Gamma_1(z)$, $\Gamma_2(z)$ are to be determined. Hence

$$\beta_x/\beta = \Gamma(z) z_x$$

where $\Gamma(z) = \Gamma_2(z)/\Gamma_1(z)$. Integrating this and using the freedom in remark 3.4(*b*) in §3,

$$\beta = \beta(t). \tag{4.4}$$

The coefficient of ww' yields

$$\beta z_x^2 z_t \Gamma_3(z) = \beta^2 \alpha z_x$$

where $\Gamma_3(z)$ is to be determined. Using (4.3*a*) and the freedom in remark 3.4(*a*), we take

$$\alpha \equiv 0. \tag{4.5}$$

Using (4.4) and (4.5), (4.2) simplifies to

$$\beta z_x^2 z_t w''' + \left(\frac{\mathrm{d}\beta}{\mathrm{d}t} z_x^2 + \beta(z_t z_{xx} + 2 z_x z_{xt})\right) w'' + \left(\frac{\mathrm{d}\beta}{\mathrm{d}t} z_{xx} + \beta(z_{xxt} + z_t + z_x)\right) w' + \frac{\mathrm{d}\beta}{\mathrm{d}t} w + \beta^3 z_x w^2 w' = 0 \tag{4.6}$$

Now suppose that $\mathrm{d}\beta/\mathrm{d}t \not\equiv 0$ and using the coefficient of $w^2 w'$ as the normalising coefficient, then the coefficient of w yields

$$\frac{1}{\beta^3} \frac{\mathrm{d}\beta}{\mathrm{d}t} = z_x \Gamma(z).$$

Integrating this and using the freedom in remark 3.4(*c*) gives

$$z(x, t) = \frac{x}{\beta^3} \frac{\mathrm{d}\beta}{\mathrm{d}t} + \sigma(t) \tag{4.7}$$

where $\sigma(t)$ is to be determined. The coefficient of w'', after using (4.7), yields

$$\Gamma_4(z) = \frac{2}{\beta^5} \frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} - \frac{5}{\beta^6} \left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)^2.$$

Therefore $\Gamma_4(z) = A$, which is a constant. Multiplying by $\mathrm{d}\beta/\mathrm{d}t$ and integrating, we have

$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)^2 = A\beta^6 + B\beta^5 \tag{4.8}$$

with B another constant. The coefficient of w''', after using (4.7), yields

$$\begin{aligned}
\frac{1}{\beta^5} \frac{\mathrm{d}\beta}{\mathrm{d}t} \left[x \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{\beta^3} \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right] &= \Gamma_5(z) \\
&= Cz + D \\
&= C\left(\frac{x}{\beta^3} \frac{\mathrm{d}\beta}{\mathrm{d}t} + \sigma(t)\right) + D
\end{aligned}$$

with C, D constants (since Γ_5 in linear in x, then necessarily it is linear in z). By integrating the coefficient of x we find that

$$\left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)^2 = 2C\beta^6 \ln\beta + E\beta^6 \tag{4.9}$$

with E a constant. Equations (4.8) and (4.9) are compatible only if $A = E$ $(= 1$, without loss of generality) and $B = C = 0$. Hence

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \beta^3$$

and so (4.6) simplifies to

$$\beta^3(w + w^2w' + w'') + \beta w' + \beta\frac{\mathrm{d}\sigma}{\mathrm{d}t}(w' + w''') = 0$$

with $z = x + \sigma(t)$. It is easily shown that this is an ordinary differential equation for $w(z)$ if and only if

$$\beta \equiv 1 \qquad \sigma(t) = \gamma_1 t + \gamma_0$$

with γ_1, γ_0 arbitrary constants. However, this contradicts the assumption that $\mathrm{d}\beta/\mathrm{d}t \not\equiv 0$.

If $\mathrm{d}\beta/\mathrm{d}t \equiv 0$, then without loss of generality we assume that $\beta \equiv 1$, and then (4.6) simplifies to

$$z_x^2 z_t w''' + (z_t z_{xx} + 2z_x z_{xt})w'' + (z_{xxt} + z_t + z_x)w' + z_x w^2 w' = 0. \tag{4.10}$$

This is an ordinary differential equation for $w(z)$ provided that

$$\Gamma_1(z) = z_x z_t \qquad z_x\Gamma_2(z) = z_{xx}z_t + 2z_x z_{xt} \qquad z_x\Gamma_3(z) = z_{xxt} + z_t$$

where $\Gamma_1(z)$, $\Gamma_2(z)$, $\Gamma_3(z)$ are to be determined. Hence

$$z_{xt}/z_t = -z_x\Gamma(z)$$

with $\Gamma = (\Gamma_2 - \Gamma_1')/\Gamma_1$. Then integrating twice and using the freedom in remark 3.4(c) yields

$$z(x,t) = \theta(x) + \sigma(t)$$

where $\theta(x)$, $\sigma(t)$ are to be determined, and so (4.10) simplifies further to

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \frac{\mathrm{d}\sigma}{\mathrm{d}t} w''' + \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\frac{\mathrm{d}\sigma}{\mathrm{d}t} w'' + \left(\frac{\mathrm{d}\theta}{\mathrm{d}x} + \frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) w + \frac{\mathrm{d}\theta}{\mathrm{d}x} w^3 = 0.$$

It is easily shown that this is an ordinary differential equation for $w(z)$ if and only if

$$\theta(x) = \gamma_1(x + x_0) \qquad \sigma(t) = \gamma_2(t + t_0)$$

where γ_1, γ_2, x_0, t_0 are arbitrary constants.

Hence necessary and sufficient conditions for the MBBM equation to have a similarity reduction in the form $u(x,t) = U(x,t,w(z))$, for some U and z, are that $u(x,t) = w(z)$, with $z = \gamma_1 x + \gamma_2 t$, where γ_1, γ_2 are constants, which is just the travelling wave solution.

Now we consider the case where $z_x z_t \equiv 0$. It is suffices to seek solutions in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)y(x).$$

Substituting this into the MBBM equation (1.8) yields

$$\alpha_t + \beta_t y + \left(1 + \alpha_x + \beta_x y + \beta y'\right)\left(\alpha^2 + 2\alpha\beta y + \beta^2 y^2\right) + \alpha_{xxt} + \beta_{xxt} y + 2\beta_{xt} y' + \beta_t y'' = 0. \tag{4.11}$$

First, it is easily shown that $\alpha(x,t) \equiv 0$, $\beta(x,t) \equiv \beta(t)$, and $\beta(t)$ satisfies the equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = c\beta^3$$

with c constant. Hence (4.11) simplifies to

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}(y + y'') + \beta(1 + \beta^2 y^2)y' = 0.$$

This is an ordinary differential equation for $y(x)$ only if $\beta = \beta_0$, a constant, and so there are no special solutions of the MBBM equation in this case.

Therefore we conclude that the only non-constant similarity reduction of the MBBM equation obtainable either using the classical Lie method or this direct method due to Clarkson and Kruskal (1989), is the travelling wave solution

$$u(x,t) = w(z) \qquad z = x - ct$$

where $w(z)$ satisfies

$$c(w')^2 = \tfrac{1}{6}w^4 + (1-c)w^2 + Aw + B$$

with A, B constants of integration. This is solvable in terms of Jacobian elliptic functions and so is of Painlevé type.

We remark that at first sight this might appear to be a negative result; however, on the contrary, by using the direct method, it has been proved that the travelling wave solution is the *only* non-constant similarity reduction of the MBBM equation of the form $u(x,t) = U(x,t,w(z(x,t)))$, where U and z are specified functions and $w(z)$ satisfies an ordinary differential equation (this being the general form of a similarity reduction, cf Bluman and Cole 1974). Previous results on the existence of similarity reductions for the MBBM equation used the classical Lie group method (Clarkson 1983, 1986), and as the SRLW equation clearly demonstrates, this method does not necessarily determine all possible similarity reductions of a given partial differential equation.

5. Painlevé analysis

In this section we apply the Painlevé tests to the SRLW and MBBM equations. The *Painlevé conjecture* (or *Painlevé ODE test*) as formulated by Ablowitz *et al* (1978, 1980) and Hastings and McLeod (1980) asserts that every ordinary differential equation which arises as a similarity reduction of a non-linear partial differential equation solvable by inverse scattering is of *Painlevé type* (i.e. its solutions have no movable singularities other than poles), though perhaps only after a transformation of variables. Ablowitz *et al* (1980) and McLeod and Olver (1983) have given proofs of the Painlevé ODE test under certain restrictions.

Subsequently, Weiss *et al* (1983) proposed the *Painlevé PDE test* as a method of applying the Painlevé ODE test directly to a given partial differential equation without having to consider similarity reductions (which might not exist anyway). A partial differential equation is said to possess the Painlevé property if all its solutions are 'single-valued' in the neighbourhood of arbitrary, non-characteristic, movable singularity manifolds. As for the Painlevé ODE test, at present there is no rigorous proof of the Painlevé PDE test, though a partial proof can be inferred from the partial proof of the Painlevé ODE test due to McLeod and Olver (1983). Despite being by no means foolproof (as we discuss in §6 below—see also Pogrebkov 1989), the Painlevé tests appear to provide useful criteria for the identification of completely integrable partial differential equations.

The method for applying the Painlevé PDE test introduced by Weiss *et al* (1983) (with simplifications due to Kruskal, private communication, 1984), involves seeking a solution of a given partial differential equation in the form

$$u(x,t) = \phi^{-p}(x,t)\sum_{j=0}^{\infty} u_j(t)\phi^j(x,t) \qquad u_0 \not\equiv 0 \tag{5.1a}$$

with

$$\phi(x,t) = x + \psi(t) = 0 \tag{5.1b}$$

where $\psi(t)$ is an arbitrary analytic function of t, and $u_j(t)$, $j = 0, 1, 2, \ldots$, are analytic functions of t, in the neighbourhood of a non-characteristic movable singularity manifold defined by $\phi = 0$ (since this is assumed to be non-characteristic, then necessarily $\phi_x \neq 0$, and so we may assume that ϕ has the special form (5.1b), without loss of generality). Substituting (5.1) into the partial differential equation and equating coefficients of like powers of ϕ determines p and defines recursion relations for u_n, for $n \geq 1$, of the form

$$(n-\beta_1)(n-\beta_2)\ldots(n-\beta_N)u_n = F_n(u_0, u_1, \ldots, u_{n-1}, \psi) \tag{5.2}$$

where N is the order of the equation, for some functional F_n. This defines u_n unless $n = \beta_j$ for some j, $1 \leq j \leq N$. $n = \beta_1, \beta_2, \ldots, \beta_N$ are the *resonances* (commonly $n = -1$ is a resonance and it is usually associated with the singularity manifold defined by $\phi = 0$ being arbitrary). For each positive integer resonance there is a compatibility condition (i.e. $F_\beta = 0$) which must be identically satisfied for the partial differential equation to have a solution of the form (5.1) and then $u_\beta(t)$ is an arbitrary function. Essentially, in order for a given partial differential equation to pass the Painlevé PDE test, it is required that p is an integer and there are $N-1$ consistent recursion relations (i.e. all the compatibility conditions are satisfied), so that the series (5.1) contains the requisite number of arbitrary functions as required by the Cauchy–Kovalevski theorem ($\psi(t)$ is the Nth arbitrary function) and thus corresponds to the general solution of the equation.

In addition to providing a valuable first test for whether a given partial differential equation is completely integrable, other important information relating to completely integrable equations can be obtained by use of Painlevé analysis, including Bäcklund transformations, Lax pairs, Hirota's bilinear representation, special and rational solutions, etc (cf Chudnovsky *et al* 1983, Gibbon *et al* 1985, 1988, Newell *et al* 1987, Weiss 1983, 1984a,b, 1985a,b, 1986a,b, 1987). Many of these results are obtained by seeking solutions of the partial differential equation in the form of a truncated Laurent series expansion

$$u(x,t) = u_0(x,t)\phi^{-p}(x,t) + u_1(x,t)\phi^{-p+1}(x,t) + \cdots + u_p(x,t). \tag{5.3}$$

We remark that for a truncated expansion, ϕ is generally *not* assumed to have the special form (5.1*b*); furthermore, equating the coefficient of each order of ϕ to zero can be too restrictive and a more general approach is required (cf Newell *et al* 1987).

If a compatibility condition is not satisfied for arbitrary ϕ (i.e. $F_\beta \not\equiv 0$ for some β), then it is necessary to introduce terms of the form $\phi^{\beta-p}\ln\phi$ into the series (5.1) at this order to make the recursion relations consistent, thereby rendering it a multivalued logarithmic psi series. However, if for special choices of ϕ, all the compatibility conditions are identically satisfied (i.e. ϕ satisfies a set of 'consistency conditions' $F_\beta = 0$), then the equation is said the have the 'conditional Painlevé property' (Weiss 1984b). In these cases, useful information, such as special solutions, for non-integrable equations can be obtained by using truncated Laurent series expansions (5.3) (cf Cariello and Tabor 1989, Conte 1988, Conte and Musette 1989, Fournier and Spiegel 1987, Nozaki 1987 and Weiss 1984b). Additionally, even if the equation possesses neither the Painlevé property nor the conditional Painlevé property for any choice of ϕ, then analysis of the associated logarithmic psi series can still yield valuable insights (cf Fournier *et al* 1988, Levine and Tabor 1988).

We also remark that symbolic manipulation programs have been developed, both in MACSYMA and REDUCE, to assist in the application of the Painlevé tests (cf Rand and Winternitz 1986, Hlavatý 1986, Hereman and van den Bulck 1988).

5.1. The SRLW equation and the Painlevé ODE test

In §3 we derived two type of similarity reductions for the SRLW equation (1.7).

Case 1. Consider the travelling wave solution

$$u(x,t) = v(z) \qquad z = x - ct \tag{5.4}$$

where $v(z)$ satisfies the ordinary differential equation

$$(c^2+1)v - \tfrac{1}{2}cv^2 + c^2v'' = Az + B \tag{5.5}$$

with $' = \mathrm{d}/\mathrm{d}z$ and A, B arbitrary constants. If $c \neq 0$, then depending upon the choice of the constants, the solution $v(z)$ of equation (5.5) is expressible in terms of either (i) if $A = 0$, elliptic functions (cf Whittaker and Watson 1927), or (ii) if $A \neq 0$, the first Painlevé equation (cf Ince 1956)

$$\frac{\mathrm{d}^2 V}{dZ^2} = 6V^2 + Z.$$

In both cases all solutions of equation (5.5) are meromorphic and so it is of Painlevé type.

Case 2. Consider the similarity reduction

$$u(x,t) = t^{-1}w(z) + \lambda t \qquad z = x - \frac{\ln t}{\lambda} \tag{5.6}$$

where λ ($\not\equiv 0$) is a constant and $w(z)$ satisfies

$$w'''' + 3\lambda w''' + (2\lambda^2 + 1)w'' + 3\lambda w' + 2\lambda^2 w - 2\lambda^2 ww' - \lambda[ww'' + (w')^2] = 0. \tag{5.7}$$

Using the algorithm developed by Ablowitz *et al* (1980), it is easily shown that in the neighbourhood of an arbitrary point z_0,

$$\begin{aligned} w(z) = {} & \frac{12}{\lambda}(z-z_0)^{-2} - \frac{12}{5}(z-z_0)^{-1} + \frac{25-\lambda^2}{25} - \left(\frac{125+\lambda^2}{125}\right)(z-z_0) \\ & + \left(w_4 - \frac{6}{25}\lambda \ln(z-z_0)\right)(z-z_0)^2 + \mathrm{o}((z-z_0)^3) \end{aligned}$$

with w_4 an arbitrary constant. At higher orders of $z - z_0$, higher and higher powers of $\ln(z - z_0)$ are required. Hence the general solution of equation (5.7) has a movable logarithmic branch point and so it is not of Painlevé type.

Since the SRLW equation (1.7) is reducible through the similarity reduction (5.6) to an ordinary differential equation (5.7) which is *not* of Painlevé type, then the Painlevé ODE test predicts that it is not solvable by inverse scattering. (We note that if we had only considered similarity reductions obtained using the classical Lie group method, then there is only one such (non-constant) similarity reduction, namely the travelling wave solution (5.4), which reduces the SRLW equation to an ordinary differential equation which is of Painlevé type.)

5.2. *The SRLW equation and the Painlevé PDE test*

In order to apply the Painlevé PDE test to the SRLW equation (1.7), we seek a solution of the equation in the form (5.1) (in the neighbourhood of an arbitrary non-characteristic, movable singularity manifold defined by $\phi = 0$). By leading-order analysis it is seen that $p = 2$, and $u_0 = -12\mathrm{d}\psi/\mathrm{d}t$. Equating coefficients of like powers of ϕ yields, for $n \geq 1$, the general recursion relation

$$\begin{aligned} & (n+1)(n-4)(n-5)(n-6)u_n\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2 + (n-4)(n-5)\left\{\frac{1}{2}\frac{\mathrm{d}\psi}{\mathrm{d}t}\sum_{j=1}^{n-1} u_j u_{n-j}\right. \\ & \qquad \left. + (n-3)\left(2\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}\frac{\mathrm{d}\psi}{\mathrm{d}t} + u_{n-1}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right) + \frac{\mathrm{d}^2 u_{n-2}}{\mathrm{d}t^2} + \left[1 + \left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^2\right]u_{n-2}\right\} \\ & \qquad + (n-5)\left[\left(2\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}\frac{\mathrm{d}\psi}{\mathrm{d}t} + u_{n-1}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=0}^{n-1} u_j u_{n-1-j}\right)\right] + \frac{\mathrm{d}^2 u_{n-4}}{\mathrm{d}t^2} \\ & = 0 \end{aligned}$$

(define $u_j = 0$ for $j < 0$). Thus the resonances are $n = -1$, 4, 5, 6. The first few recursion relations yield

$$u_1 = \frac{12}{5}\frac{d^2\psi}{dt^2}\left(\frac{d\psi}{dt}\right)^{-1} \tag{5.8a}$$

$$u_2 = \left[\frac{11}{25}\left(\frac{d^2\psi}{dt^2}\right)^2 - \frac{1}{5}\frac{d\psi}{dt}\frac{d^3\psi}{dt^3} - \left(\frac{d\psi}{dt}\right)^2 - \left(\frac{d\psi}{dt}\right)^4\right]\left(\frac{d\psi}{dt}\right)^{-3} \tag{5.8b}$$

$$u_3 = \left[\frac{11}{125}\left(\frac{d^2\psi}{dt^2}\right)^3 - \frac{1}{25}\left(\frac{d\psi}{dt}\right)^2\frac{d^2\psi}{dt^2}\frac{d^3\psi}{dt^3} + \left(\frac{d\psi}{dt}\right)^4\frac{d^2\psi}{dt^2}\right]\left(\frac{d\psi}{dt}\right)^{-5} \tag{5.8c}$$

and the compatibility conditions

$$\frac{d}{dt}\left(12u_3\frac{d\psi}{dt} - u_1u_2\right) = 2\frac{du_1}{dt}\frac{d\psi}{dt} + u_1\frac{d^2\psi}{dt^2} + 12\frac{d^3\psi}{dt^3} \tag{5.9a}$$

$$\frac{d^2u_1}{dt^2} = 0 \tag{5.9b}$$

$$\begin{aligned} 6\frac{du_5}{dt}\frac{d\psi}{dt} &+ 2\frac{d^2u_4}{dt^2} + (2u_1u_5 + 2u_2u_4 + u_3^2)\frac{d\psi}{dt} + \frac{d}{dt}(u_1u_4 + u_2u_3) \\ &+ 2u_4\left[1 + \left(\frac{d\psi}{dt}\right)^2\right] + 2\frac{du_3}{dt}\frac{d\psi}{dt} + u_3\frac{d^2\psi}{dt^2} + \frac{d^2u_2}{dt^2} = 0. \end{aligned} \tag{5.9c}$$

Using (5.8), the compatibility conditions (5.9a, b) simplify to

$$\frac{d\psi}{dt}\frac{d^3\psi}{dt^3} - 2\left(\frac{d^2\psi}{dt^2}\right)^2 = \left(\frac{d\psi}{dt}\right)^3\frac{d^3\psi}{dt^3} - \left(\frac{d\psi}{dt}\frac{d^2\psi}{dt^2}\right)^2 \tag{5.10a}$$

$$\frac{d^2}{dt^2}\left[\frac{d^2\psi}{dt^2}\left(\frac{d\psi}{dt}\right)^{-1}\right] = 0 \tag{5.10b}$$

which clearly are not identically satisfied for arbitrary $\psi(t)$. Therefore, in general, it is necessary to introduce a logarithmic term of the form $u_{4,1}(t)\phi^2 \ln \phi$, into the series (5.1) at this order, and so it becomes a multivalued logarithmic psi series (at higher orders of ϕ, higher and higher powers of $\ln \phi$ are required). Such an expansion indicates that the SRLW equation does not pass the Painlevé PDE test.

It is easily shown that the only solution of equations (5.10) is

$$\psi(t) = -ct + k \tag{5.11}$$

with c, k arbitrary constants, thus $\phi(x, t) = x - ct + k$, which is the singularity manifold for a travelling wave solution. In this case the coefficients u_n in the Laurent series (5.1) are constants and from equations (5.8) and (5.9) we see that

$$u_0 = 12c \qquad u_1 = 0 \qquad u_2 = (c^2 + 1)/c \qquad u_3 = 0.$$

For these, all the compatibility conditions (5.9) are identically satisfied and so there is no need for logarithmic terms to be introduced into the Laurent series expansion. (This is because the travelling wave solution reduces the SRLW equation (1.7) to equation (5.5), which is solvable in terms of either elliptic functions or the first Painlevé equation, all solutions of which are meromorphic.) Therefore (5.11) is a necessary and sufficient condition for the absence of logarithmic terms in the Laurent series expansion (5.1).

5.3. *The* MBBM *equation and the Painlevé* ODE *test*

The only non-constant similarity reduction of the MBBM equation (1.8) is the travelling wave solution $u(x,t) = w(z)$, where $z = x - ct$ and $w(z)$ satisfies

$$c(w')^2 = \tfrac{1}{6}w^4 + (1-c)w^2 + Aw + B \tag{5.12}$$

with A, B constants of integration. This is solvable in terms of Jacobian elliptic functions and so is of Painlevé type. Therefore every ordinary differential equation which arises as a similarity reduction of the MBBM equation (1.8) is of Painlevé type, and so the MBBM equation satisfies the necessary conditions of the Painlevé ODE test to be solvable by inverse scattering.

5.4. *The* MBBM *equation and the Painlevé* PDE *test*

To apply the Painlevé PDE test to the MBBM equation (1.8), we seek a solution of the equation in the form (5.1). By leading-order analysis it is easily seen that $p = 1$, and $u_0^2 = -6\mathrm{d}\psi/\mathrm{d}t$. Equating coefficients of like powers of ϕ yields, for $n \geq 1$, the general recursion relation

$$\begin{aligned}(n+1)(n-3)(n-4)u_n\frac{\mathrm{d}\psi}{\mathrm{d}t} &+ (n-2)(n-3)\frac{\mathrm{d}u_{n-1}}{\mathrm{d}t}\frac{\mathrm{d}u_{n-3}}{\mathrm{d}t} \\ &+ (n-3)\left[\frac{1}{3}\sum_{k=1}^{n-1}\sum_{j=0}^{k}u_ju_{k-j}u_{n-k} + \frac{1}{3}u_0\sum_{j=1}^{n-1}u_ju_{n-j} + \left(1+\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)u_{n-2}\right] = 0\end{aligned}$$

(define $u_j = 0$ for $j < 0$). Thus the resonances are $n = -1, 3, 4$. The first few recursion relations yield

$$u_1 = \frac{\mathrm{i}}{2\sqrt{6}}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^{-3/2} \tag{5.13a}$$

$$u_2 = -\frac{\mathrm{i}}{\sqrt{6}}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^{-7/2}\left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^3\left(1+\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right) - \frac{1}{24}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}\right)^2\right] \tag{5.13b}$$

and the compatibility conditions

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = 0. \tag{5.14a}$$

$$2\frac{\mathrm{d}u_3}{\mathrm{d}t} + 2u_0u_1u_3 + u_0u_2^2 + u_1^2u_2 + \frac{\mathrm{d}u_1}{\mathrm{d}t} + \left(1+\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)u_2 = 0. \tag{5.14b}$$

Using (5.20), we see that the compatibility condition (5.14a) is not identically satisfied unless

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} \equiv 0 \tag{5.15}$$

which clearly does not hold for arbitrary $\psi(t)$ and, in general, it is necessary to introduce a logarithmic term into the expansion (5.1) at this order. Such an expansion indicates that the MBBM equation does not pass the Painlevé PDE test.

For travelling wave solutions, $\psi(t) = -ct + k$, with c, k constants, which is the general solution of (5.15). Furthermore, the coefficients u_n in the Laurent series (5.1) are constants, and

$$u_0 = \sqrt{6c} \qquad u_1 = 0 \qquad u_2 = \frac{c-1}{\sqrt{6c}}.$$

For these constants, it is easily shown that compatibility conditions (5.22) are both identically satisfied and so there is no need for logarithmic terms to be introduced into the expansion. (This is because the travelling wave solution reduces the MBBM equation (1.8) to equation (5.12), all solutions of which are meromorphic.) Therefore (5.15) is a necessary and sufficient condition for the absense of logarithmic terms in the Laurent series expansion.

6. Discussion

First we make some general remarks about similarity reductions of partial differential equations. Generally, given a partial differential equation with a symmetry (i.e. a transformation of the dependent and/or independent variables that leaves the equation invariant), the action of the symmetry group takes one solution of the equation into another solution of the equation. Starting with a fixed solution that corresponds to the identity element of the group, then every element of the group corresponds to some solution of the (same) equation—the starting solution can be *any* solution of the equation. This mapping can be used to *define* a symmetry, and the group carries the set of all solutions of the partial differential equation into itself. Given such a symmetry of a partial differential equation, one can seek solutions which are mapped into themselves under the action of the group. These are *similarity reductions* corresponding to the group. For a partial differential equation with two independent and one dependent variables, these solutions typically are solutions of an ordinary differential equation.

Alternatively, the ordinary differential equation can be taken as a means of generating similarity reductions (or special solutions) of the partial differential equation, without regard to what maps into what. Then the ordinary differential equation appears to be an example of the *side condition* introduced by Olver and Rosenau (1986, 1987). This seems to be the way that similarity reductions are generally used. The similarity reductions obtained here are defined through an ordinary differential equation that is 'compatible' with the partial differential equation (in the sense that they have common solutions). Again, the ordinary differential equation is a side condition on the partial differential equation, and the surprise is that there exist common solutions. The issue of mapping solutions of the partial differential equation does not arise in the procedure used in this paper and so there is no connection with remarks in the previous paragraph (in fact, as shown below, the transformation groups associated with the additional similarity reductions do *not* map solutions of the partial differential equation into itself). This direct method appears to be an effective procedure of generating similarity reductions of given partial differential equations.

A one-parameter (γ) group associated with the similarity reduction

$$u(x,t) = t^{-1}w(z) + \lambda t \qquad z = x - \frac{\ln t}{\lambda} \tag{6.1}$$

is

$$(x, t, u) \to \left(x + \gamma/\lambda, t\,\mathrm{e}^{\gamma}, u\,\mathrm{e}^{-\gamma} + \lambda t(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma})\right). \tag{6.2}$$

This group maps solutions of the SRLW equation (1.7) into solutions of the 'perturbed SRLW equation'

$$u_{tt} + u_{xx} + uu_{xt} + u_x u_t + u_{xxtt} = (1 - \mathrm{e}^{-2\gamma})\Phi(x, t, u) \tag{6.3a}$$

$$\Phi(x, t, u) := u_{xx} + \lambda t u_{xt} + \lambda u_x \equiv 0. \tag{6.3b}$$

Additionally, if $u(x, t)$ is the similarity reduction (6.1), then it is easily seen that $\Phi(x, t, u) \equiv 0$. Although the group (6.2) does *not* map solutions of the SRLW equation into itself, it does give rise to similarity reductions since (6.1) identically satisfies the perturbed part of the equation. This poses the question: what type of 'symmetries' of the SRLW equation (1.7) are those we have obtained, which are not found using the classical Lie method? (They are 'weak-symmetries' in the terminology of Olver and Rosenau (1986,1987).)

In order to understand why the perturbation $\Phi(x, t, u)$ vanishes identically, recall that the infinitesimals for the similarity reduction (6.1) are

$$X(x, t, u) = \lambda^{-1} \qquad T(x, t, u) = t \qquad U(x, t, u) = 2\lambda t - u.$$

These necessarily satisfy the invariant surface condition

$$X(x, t, u)u_x + T(x, t, u)u_t - U(x, t, u) = 0 \tag{6.4}$$

i.e.

$$\psi := u_x - \lambda t u_t - 2\lambda^2 t + \lambda u = 0.$$

It is easily seen that $\Phi = \psi_x$, so $\psi \equiv 0$ implies that $\Phi \equiv 0$ (but not conversely). It appears that this observation provides an insight as to why there exist partial differential equations which possess similarity reductions whose associated groups do not map solutions of the partial differential equation into itself.

All the similarity reductions obtained in this paper are Lie point transformations, since the infinitesimals depend only on the independent variables x, t and the dependent variable u, but not upon the derivatives of u. (If the transformations also depend upon the derivative of the dependent variable, then the associated symmetries are known as *Lie–Bäcklund* symmetries, and are also determined by an algorithmic method—see Anderson and Ibragimov 1979, Ibragimov 1985, Olver 1986.) An open question which this direct method of determining similarity reductions poses is: what is the relationship (if any) between this method, and other generalisations of the classical Lie method, in particular to those due to Bluman and Cole (1969), Olver and Rosenau (1986, 1987) and Bluman *et al* (1988)? It is my opinion that the similarity reduction (6.1) should be obtainable using the non-classical method due to Bluman and Cole (1969). It is known that the similarity reductions of the Boussinesq equation (1.6) which are derived using this direct method (Clarkson and Kruskal 1989), can also be derived using Bluman and Cole's non-classical method (Levi and Winternitz 1989). However, even if the new similarity reductions derived here are theoretically obtainable by any of these generalisations of the classical Lie group method, it seems that the direct method is somewhat simpler to implement; in fact, it appears to be simpler even than the classical Lie group method for some equations (without the assistance of a symbolic manipulation program).

In Bluman and Cole's non-classical method, one utilises both the partial differential equation and the invariance surface condition (6.4), which describes a relationship between the infinitesimals X, T, U and so there are really only *two* independent infinitesimals. Assuming that $T \not\equiv 0$, we set $T = 1$, without loss of generality, and so the invariance surface condition (6.4) becomes

$$u_t = U(x, t, u) - X(x, t, u)u_x.$$

Now in the expressions for the infinitesimals U^x, U^t, etc (recall equations (2.2)), one replaces u_t by $U - u_x$ (and analogously for u_{xt}, u_{tt}, etc). Then proceeding in a similar manner to the classical Lie group method, one determines the infinitesimals $X(x, t, u)$ and $U(x, t, u)$ by collecting coefficients of like x-derivatives of u and equating them to zero. However, in the classical Lie group method, one obtains a *linear, homogeneous* system of determining equations (cf Bluman and Cole 1974), whereas in the non-classical method, the system of determining equations is usually *non-linear*. In fact, Olver and Rosenau (1987), suggest that for some partial differential equations, these determining equations for the non-classical method might be actually too difficult to explicitly solve.

There are two observations which suggest that there exists a relationship between the direct method developed by Clarkson and Kruskal (1989) and the non-classical method due to Bluman and Cole (1969):

(a) both methods involve solving an overdetermined system of non-linear equations;

(b) the invariance surface condition (6.4) appears to play a central role.

In their generalisation of Bluman and Cole's non-classical method (1969), Olver and Rosenau (1986, 1987) show that in order to determine a group-invariant solution of a given partial differential equation, one can use *any* group of infinitesimal transformations. However, in general, given any group of infinitesimal transformations and any partial differential equation, there will be *no* solutions invariant under the group and so the question becomes how does one determine *a priori* whether a given group will give a meaningful similarity reduction? One possibility is that by seeking a solution in a certain form (as we have done in this paper), one is naturally led to the appropriate group (i.e. the requirement that the similarity reduction reduces the partial differential equation to an ordinary differential equation is equivalent to the 'side conditions' in the terminology of Olver and Rosenau 1986, 1987). The results obtained both here and in an earlier papers (Clarkson 1989, Clarkson and Kruskal 1989) support the conclusions drawn by Olver and Rosenau (1986) that 'the unifying theme behind finding special solutions of partial differential equations is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations'. Nevertheless, group theory clearly remains important in the determination of explicit, physically significant, special solutions of partial differential equations (as also demonstrated by Olver and Rosenau 1987).

Now we shall make some remarks on the Painlevé tests. In the literature there has been much lively debate as to whether the Painlevé tests provide necessary and/or sufficient conditions for a given partial differential equation to be completely integrable. As described above, the Painlevé tests are *necessary* conditions; however, some authors (e.g. Weiss *et al* 1983) interpret them as *sufficient* conditions.

A prime source of contention concerns quasilinear partial differential equations such as the Dym equation (Kruskal 1975)

$$u_t = (u^{-1/2})_{xxx} \tag{6.5}$$

which is solvable by inverse scattering (Wadati *et al* 1979—see also Calogero and Degasperis 1982). The Dym equation can be transformed via hodograph transformations (i.e. transformations involving the interchange of dependent and independent variables), into both the KdV equation (1.4) (cf Levi *et al* 1984), and the MKdV equation (1.5) (cf Kawamoto 1985), both of which are solvable by inverse scattering and pass the Painlevé PDE test (Weiss *et al* 1983). However, the Dym equation (6.5) does not *directly* pass the Painlevé PDE test since it has an expansion of the form

$$u(x,t) = \phi^{-4/3}(x,t) \sum_{j=0}^{\infty} u_j(t)\phi^{j/3}(x,t)$$

with $\phi(x,t) = x+\psi(t)$, in the neighbourhood of an arbitrary non-characteristic movable singularity manifold defined by $\phi = 0$ (Weiss 1983) and so it is 'weak-Painlevé' (cf Ramani *et al* 1982, Ranada *et al* 1985). Consequently, it might be conjectured that the 'weak-Painlevé' property would provide the requisite requirement, but this is not sufficient. For example, the higher KdV equation

$$u_t + u^3 u_x + u_{xxx} = 0$$

is also 'weak-Painlevé' (Weiss 1986a), yet it is thought not to be completely integrable since (a) it has only three independent polynomial conservation laws of a certain type (Miura 1976), (b) the interaction of solitary wave solutions is inelastic (Fornberg and Whitham 1978), and (c) it appears not to be solvable by inverse scattering (McLeod and Olver 1983). Therefore the 'weak-Painlevé' concept does not appear to distinguish between integrable and non-integrable partial differential equations.

Recall that the Painlevé tests require that an integrable partial differential equation possess the Painlevé property possibly only after a transformation of variables, so that we may first have to make a change of variables before applying the tests. An open question remains as to what kind of transformations are allowable in the application of the Painlevé tests (i.e. which transformations does one have to check?). It seems that completely integrable quasilinear partial differential equations such as the Dym equation (6.5), which are 'weak-Painlevé' can be transformed into a partial differential equation with the 'full-Painlevé' property through an appropriate hodograph transformation (see Clarkson and Cosgrove (1986) and Clarkson *et al* (1989) for further examples and an algorithmic method for transforming a quasilinear partial differential equation into a form seemingly more suitable for applying the Painlevé tests).

The only non-constant similarity reduction of the MBBM equation (1.8) obtained using either the classical Lie group method or the direct method, is the travelling wave solution $u(x,t) = w(x-ct)$, where $w(z)$ satisfies (5.12), which is solvable in terms of elliptic functions and so is of Painlevé type. Therefore the MBBM equation satisfies the *necessary* conditions of the Painlevé ODE test to be solvable by inverse scattering. Yet, the MBBM equation does not pass the Painlevé PDE test, strongly suggesting that it is not solvable by inverse scattering (in agreement with the numerical evidence that the interaction of solitary waves is inelastic and so are not solitons (Makhankov 1978)).

Hence we conclude that the Painlevé ODE test may *not* be generalised to provide a necessary *and sufficient* condition for a given partial differential equation to be solvable by inverse scattering. (Note that this result does *not* contradict the original Painlevé ODE test as formulated by Ablowitz *et al* (1978).) Therefore it seems that the Painlevé PDE test might provide a better criterion for the identification of completely integrable partial differential equations, though we caution that it is not always sufficient to just seek solutions in the form of the Laurent series (5.1) (cf Clarkson 1985, Weiss 1989).

For the SRLW equation (1.7), had we only considered those similarity reductions which can be obtained using the classical Lie group method, then we would have obtained a similar result. However, using the direct method, we obtained the similarity reduction (6.1) which reduces the SRLW equation to an ordinary differential equation which is not of Painlevé type, and so the Painlevé ODE test predicts that the SRLW equation is *not* solvable by inverse scattering. This conclusion is supported by the facts that (a) it does not pass the Painlevé PDE test, (b) numerical evidence showing that the interaction of solitary waves is inelastic (Bogolubsky 1977, Makhankov 1978, Seyler and Fenstermacher 1984), and (c) it possesses only three independent polynomial conservation laws (Seyler and Fenstermacher 1984).

There is much current interest in the mathematically and physically relevant determination of similarity reductions of partial differential equations (either integrable equations or, more particularly, non-integrable equations), which reduce the equations to ordinary differential equations. Painlevé analysis is frequently used to determine whether the resulting ordinary differential equation is of Painlevé type. It appears to be the case that whenever the resulting ordinary differential equation is of Painlevé type, then one can explicitly solve the ordinary differential equation and obtain exact solutions to the original partial differential equation; however if the ordinary differential equation is not of Painlevé type, then usually one cannot solve it explicitly (cf Gagnon *et al* 1989, Gagnon and Winternitz 1988, 1989a, b, c, Grundland *et al* 1987, Skierski *et al* 1988, Winternitz *et al* 1987, 1988). It appears that the use of similarity reductions in conjunction with Painlevé analysis will continue to attract much interest.

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